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A Recurrence Relation for the Square Root

W. G. HWANG

Center for Applied Mathematics, Cornell University, Ithaca, New York 14850

AND

JOHN TODD*

Department of Mathematics, California Institute of Technology, Pasadena, California 91109

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The behavior of the sequence $x_{n+1} = x_n(3N - x_n^2)/2N$ is studied for $N > 0$ and varying real x_0. When $0 < x_0 < (3N)^{1/2}$ the sequence converges quadratically to $N^{1/2}$. When $x_0 > (5N)^{1/2}$ the sequence oscillates infinitely. There is an increasing sequence β_r with $\beta_{-1} = (3N)^{1/2}$ which converges to $(5N)^{1/2}$ and is such that when $\beta_r < x_0 < \beta_{r+1}$ the sequence $\{x_n\}$ converges to $(-1)^r N^{1/2}$. For $x_0 = 0, \beta_{-1}, \beta_0, \ldots$ the sequence converges to 0. For $x_0 = (5N)^{1/2}$ the sequence oscillates: $x_n = (-1)^n(5N)^{1/2}$. The behavior for negative x_0 is obtained by symmetry.

1. INTRODUCTION

The recurrence relation

$$x_{n+1} \equiv f(x_n) = x_n(3N - x_n^2)/2N, \tag{1}$$

which converges quadratically to $N^{1/2}$—we shall assume $N > 0$—was popular in the days of computers without division instructions since it did not involve division by a variable quantity as does the Newton–Raphson relation

$$y_{n+1} \equiv g(y_n) = (y_n + (N/y_n))/2, \tag{2}$$

although (1) is slightly less rapidly convergent than (2). In fact if $\epsilon_n = x_n - N^{1/2}$ we have

$$\epsilon_{n+1} = -\epsilon_n^2(x_n + 2N^{1/2})/2N \sim -\epsilon_n^2(3/(2N^{1/2})),$$

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while if $\eta_n = y_n - N^{1/2}$ we have

$$\eta_{n+1} = \eta_n^2/(2y_n) \sim \eta_n^2(1/(2N^{1/2})).$$

It is well known that the behavior of the iterates of polynomial or rational functions can be quite complicated. (See, e.g., Chaundy and Phillips [2] for the behavior in the case $x_{n+1} = ax_n^2 + bx_n + c$ and, for more general results, De Bruijn [1] and Montel [4]. Stein and Ulam [6] and Metropolis *et al.* [3] discuss higher dimensional nonlinear transformations, partly experimentally.)

The behavior in the case of (2) is simple: If $0 < x_0 < N^{1/2}$ then $x_1 > N^{1/2}$ and $\{x_n\}_1^\infty$ decreases to $N^{1/2}$. If $x_0 = N^{1/2}$ then $x_n \equiv N^{1/2}$. If $N^{1/2} < x_0$ then $\{x_n\}_0^\infty$ decreases to $N^{1/2}$. If x_0 is negative we get convergence to $-N^{1/2}$, g being odd.

The behavior in the case of (1), summarized in the abstract, is moderately complicated. In our discussion of (1), since $f(x)$ is odd, we may restrict our attention to the case when x_0 is positive.

2. The Case When $0 \leqslant x_0 \leqslant (3N)^{1/2}$

If $x_0 = 0$ we have $x_n \equiv 0$ and $x_n \to 0$. If $x_0 = (3N)^{1/2}$ then $x_1 = x_2 = \cdots = 0$ and we have $x_n \to 0$. If $x_0 = N^{1/2}$ then $x_n \equiv N^{1/2}$. We need to deal with $0 < x_0 < (3N)^{1/2}$ only.

Note that if the sequence $x_n \to l$ then we must have

$$l = l(3N - l^2)/2N$$

so that

$$l = 0 \qquad \text{or} \qquad l = \pm N^{1/2}. \tag{3}$$

We have

$$\begin{aligned} x_{n+1} - N^{1/2} &= [x_n(3N - x_n^2) - 2N^{3/2}]/2N \\ &= -(x_n - N^{1/2})^2(x_n + 2N^{1/2})/2N. \end{aligned} \tag{4}$$

This shows that convergence will be quadratic. Moreover, if x_n is positive, $x_{n+1} - N^{1/2}$ is negative, i.e., $x_{n+1} < N^{1/2}$. Again

$$x_{n+1} - x_n = x_n(N - x_n^2)/2N, \tag{5}$$

which is positive if $0 < x_n < N^{1/2}$, negative if $x_n > N^{1/2}$. If $0 < x_0 < (3N)^{1/2}$ then $x_1 > 0$ from (1) and from (4) we conclude that $x_1 < N^{1/2}$. It follows from (5) that $x_1 < x_2 < \cdots$. Since we have already noted that $x_{n+1} < N^{1/2}$ it follows that $x_n \to N^{1/2}$.

These cases are illustrated in Fig. 1. If $x_0 = A$, then $x_1 = B$, $x_2 = C$ while if $x_0 = D$, then $x_1 = B$, $x_2 = C$.

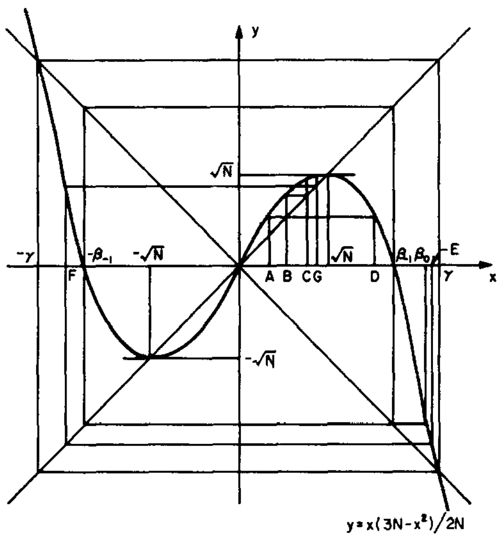

FIGURE 1

3. THE CASE WHEN $\gamma = (5N)^{1/2} \leqslant x_0$

From (1), if $x_n = \pm\gamma$, then $x_{n+1} = \mp\gamma$. Hence, when $x_0 = \gamma$, we have $x_n = (-1)^n\gamma$ and the sequence oscillates finitely.

Take $x_0 > \gamma$. Then from (1)

$$x_1 + x_0 = x_0(5N - x_0^2)/2N < 0 \tag{7}$$

so that $x_1 < -\gamma$. Also, since x_1 is negative and $x_1^2 > 5N$,

$$x_2 + x_1 = x_1(5N - x_1^2)\,2N > 0. \tag{8}$$

Subtracting (7) from (8) we find

$$x_2 - x_0 > 0. \tag{9}$$

Hence, the subsequence $\{x_{2n}\}$ is increasing; similarly, the subsequence $\{x_{2n-1}\}$ is decreasing. We shall show that neither can have a finite limit, and so the sequence oscillates infinitely.

From (1) we find

$$x_{n+2} = x_n(3N - x_n^2)(12N^3 - x_n^2(3N - x_n^2)^2)/16N^4. \tag{10}$$

If the subsequence $\{x_{2n}\}$ has a finite limit l then we must have

$$16N^4 l = l(3N - l^2)(12N^3 - l^2(9N^2 - 6Nl^2 + l^4)),$$

which gives

$$l^9 - 9Nl^7 + 27N^2l^5 - 39N^3l^3 + 20N^4 l = 0.$$

The same is true for the subsequence $\{x_{2n-1}\}$.

This equation has five real roots

$$l = 0, \qquad l = \pm N^{1/2}, \qquad l = \pm(5N)^{1/2} \tag{11}$$

and four complex ones which are the roots of

$$l^4 - 3Nl^2 + 4N^2 = 0.$$

Since, from (9), the absolute values of all the x_n exceed $x_0 > \gamma$, none of the limits in (11) is possible. Hence, the sequence oscillates infinitely.

4. The Case when $(3N)^{1/2} < x_0 < (5N)^{1/2}$

This is where the complications are. We shall show that there is an increasing sequence of numbers β_r, $r = -1, 0, 1, \ldots$ satisfying $(3N)^{1/2} \leqslant \beta_r < (5N)^{1/2}$ and such that $\beta_r \to (5N)^{1/2}$ which have the following properties:

$$\text{if } x_0 = \beta_r \text{ then } x_n \to 0 \text{ and in fact } x_{r+2} = x_{r+3} = \cdots = 0. \tag{12}$$

$$\text{if } \beta_r < x_0 < \beta_{r+1} \text{ then } x_n \to (-1)^r N^{1/2}. \tag{13}$$

We begin by discussing the equation

$$h(x) \equiv 3x^3 - 3x - 2\theta = 0, \tag{14}$$

which we can also write as

$$x(3x^2 - 5) + 2(x - \theta) = 0, \tag{15}$$

when $1 \leqslant \theta < (5/3)^{1/2} \doteqdot 1.2910$. The graph of $h(x)$ has the following form:

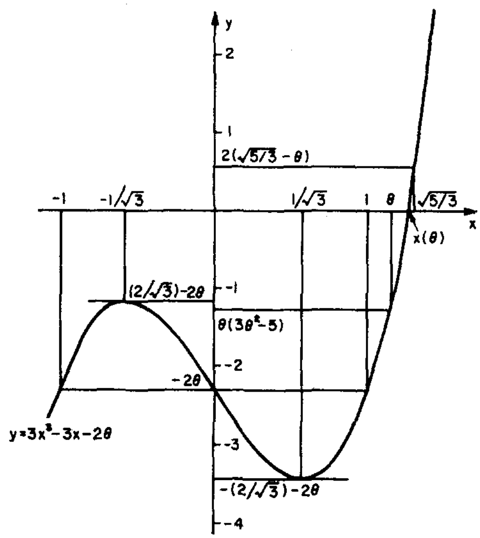

FIGURE 2

It is clear from the graph that for any such θ the equation (14) has exactly one real root $x(\theta)$ which satisfies $\theta < x(\theta) < (5/3)^{1/2}$.

We define a sequence α_n, $n = -1, 0, 1, \ldots$, as follows: $\alpha_{-1} = 1$ and for $n = -1, 0, 1, 2, \ldots$, α_{n+1} is the unique real root of

$$3x^3 - 3x - 2\alpha_n = 0. \tag{16}$$

Clearly α_n is an increasing sequence bounded above by $(5/3)^{1/2}$. Its limit l, therefore, satisfies

$$3l^3 - 3l - 2l = 0$$

and is, therefore, $(5/3)^{1/2}$. Convergence is ultimately geometric with a common ratio of $1/6$ for we have

$$\frac{\alpha_{n+1} - l}{\alpha_n - l} = \frac{2}{3[\alpha_{n+1}^2 + \alpha_{n+1} l + l^2 - 1]} \approx \frac{1}{6}.$$

We can find the early α's by use of a computer, or from the tables of Salzer *et al.* [5]:

$$\alpha_{-1} = 1, \qquad \alpha_0 = 1.2600, \qquad \alpha_1 = 1.2826, \qquad \alpha_2 = 1.2896, \ldots$$
$$\lim \alpha_n = 1.2910.$$

We now define

$$\beta_n = \alpha_n(3N)^{1/2}, \qquad n = -1, 0, 1,\dots .$$

This means that Eq. (16), which defines α_{n+1},

$$3\alpha_{n+1}^3 - 3\alpha_{n+1} - 2\alpha_n = 0, \tag{17}$$

can be rewritten as

$$\beta_{n+1}^3 - 3\beta_{n+1}N = 2N\beta_n, \tag{18}$$

i.e.,

$$\beta_n = -f(\beta_{n+1}). \tag{19}$$

We shall establish the results (12) and (13) for $r = -1, 0, 1,\dots$ by induction; however, reference to the diagrams will be helpful. For instance, if $\beta_0 < x_0 = E < \beta_1$, then $-\beta_0 < x_1 = f(x_0) = F < -\beta_1$ and $0 < x_2 = f(x_1) = G < \beta_{-1}$.

(a) If $x_0 = \beta_{-1} = (3N)^{1/2}$, then, as already noted in Section 2, we have $x_1 = x_2 = \cdots = 0$. Assume that (12) holds. Take $x_0 = \beta_{r+1}$. Then $x_1 = f(x_0) = f(\beta_{r+1}) = -\beta_r$ by (19). The induction hypothesis applies and gives $x_{r+3} = x_{r+4} = \cdots = 0$.

(b) If $\beta_{-1} < x_0 < \beta_0$, then $0 > x_1 > f(\beta_0) = -\beta_{-1}$. Hence, since $f(x)$ is odd, we have from Section 2 that $x_n \to -N^{1/2}$. Now assume (13) holds. Take x_0 such that $\beta_{r+1} < x_0 < \beta_{r+2}$. Then

$$f(\beta_{r+2}) < x_1 = f(x_0) < f(\beta_{r+1}),$$

i.e., by (19),

$$-\beta_{r+1} < x_1 < -\beta_r \qquad \text{or} \qquad \beta_r < -x_1 < \beta_{r+1}.$$

The induction hypothesis applied to x_1 gives the limit $(-1)^r N^{1/2}$ and the fact that f is odd shows that for $-x_1$ the limit is

$$(-1) \times (-1)^r N^{1/2} = (-1)^{r+1} N^{1/2}$$

as required.

5. Another Recurrence Relation

The behavior of the recurrence relation

$$y_{n+1} = 2y_n^3/[3y_n^2 - N], \tag{20}$$

which also converges quadratically to $N^{1/2}$ can be discussed similarly, or it can be read off from our results by observing that putting $x_n = N/y_n$ in (1) gives (20).

6. The Origin of the Relations

All three relations are obtainable from Newton's formula,

$$x_{n+1} = x_n - [f(x_n)/f'(x_n)],$$

for suitable f. Specifically:

$$f(x) = 1 - Nx^{-2} \quad \text{gives} \quad (1),$$

$$f(x) = N - x^2 \quad \text{gives} \quad (2),$$

$$f(x) = x^3 - Nx \quad \text{gives} \quad (20).$$

7. Optimal Starting Approximations

There has recently been considerable activity in the discussion of optimal starting values for square root algorithms. In the present context this means determining a polynomial $S(N)$ of assigned degree such that if we take $x_0 = S(N)$ then the algorithm for $N^{1/2}$ is optimal in an appropriate sense. In particular Wilson [7] discusses this problem for the algorithm

$$w_{n+1} = w_n(3/2 - (N/2)\, w_n^2), \tag{1'}$$

which converges to $N^{-1/2}$, from which $N^{1/2}$ can be obtained with one multiplication. Wilson notes that the optimal polynomials for (1) are got by reversing those he obtained for (1′). Wilson does not discuss the regions of convergence.

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